Regge symmetry of 6-j or super 6-jS symbols: a re-analysis with partition properties

Lionel Bréhamet

Regge symmetry of 6-j or super 6-jS symbols: a re-analysis with partition properties

Lionel Bréhamet [‡]
Research scientist
France

Abstract

It is shown that the five Regge transformations act as a spectrometric splitter on any 6-j symbol. Four unknown partitions are brought out: S4(0), S4(1), S(2) and S4(5). They are stable subsets, with well defined parameters depending only on triangles and quadrangles. These findings are easily generalized to super 6-jS symbols, properly labelled by their own parity alpha, beta, gamma. Super Regge symmetry is reduced only for beta where S4(2), S4(5) vanish. In addition, all tools for computing exact values of any 6-jS are provided.

PACS: 03.65.Fd Algebraic methods
PACS: 02.20-a Group theory
PACS: 11.30.Pb Supersymmetry
Keywords: Angular momentum in Quantum Mechanics, 6-j symbols, Regge symmetry.

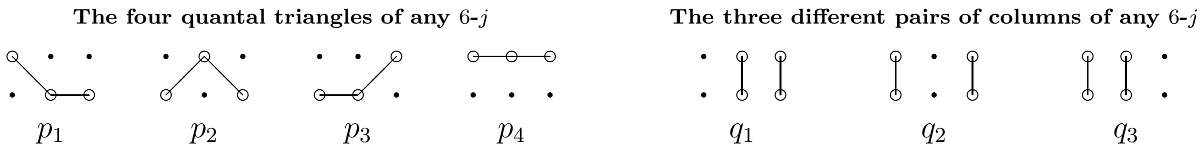
[‡]email: brehamet.l@orange.fr

1 Introduction

Thanks to Regge [1], new symmetries became known since 1959 and relate to Racah-Wigner n-j symbols. This is quite well referenced in standard books on Quantum Mechanics and Angular Momentum, like [2]. These symmetries generate new quantal triangles whose features have `never` been fully investigated as will be done in this paper.

2 Recalls about 6-j or 6-j^S symbol

Their numerical values and other properties are determined by seven non independent parameters p_1, p_2, p_3, p_4 (triangles) and q_1, q_2, q_3 (quadrangles) defined below:



The notations adopted are: $p_i, i = [1,4]$ and $q_k, k = [1,3]$, where p_i is the sum of the values of the three circled spins just above p_i in the diagrams. In the same way, q_k is the sum of the values of the four circled spins above q_k.

2.1 Analytic formula for standard 6-j symbols

Expression of a 6-j symbol is well known [3, p. 99], however we leave out the traditional notation with triangles $\triangle$ for adopting here a `re-transcription in terms of` p_i, q_k:

$$\begin{Bmatrix} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{Bmatrix} = \left[\frac{\prod_{k=1}^{k=3} \prod_{i=1}^{i=4} (q_k - p_i)!}{\prod_{i=1}^{i=4} (p_i + 1)!} \right]^{\frac{1}{2}} \sum_z \frac{(-1)^z (z+1)!}{\prod_{i=1}^{i=4} (z-p_i)! \prod_{k=1}^{k=3} (q_k - z)!} \,. \tag{2.1a}$$

2.2 Analytic formula for super 6-j^S symbols

Extension from $so(3)$ to super algebra $osp(1|2)$ was finalized in a paper dated 1993 by Daumens *et al.* [4] . Then a standard 6-j symbol becomes a super 6-j^S symbol denoted by $\begin{Bmatrix} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{Bmatrix}^S$. Essential Racah-Wigner calculus properties were studied.

`However` no analytic formula was brought out for expressing a super $\{6\text{-}j\}^S$ symbol as a single summation over an integer z like for $so(3)$.
In 2006 we succeeded in derive such a formula [5]. It was found that any $\{6\text{-}j\}^S$ should be labelled by an additional parity parameter $\pi{=}\alpha, \beta, \gamma$ according to new properties of these symbols namely: triangles p_i can be integer or half-integer. A right notation will be $\{6\text{-}j\}^S_\pi$.
Let us recall our definitions :

$$\begin{cases} \pi = \alpha & \text{if} \quad \forall i \in [1,4] \quad p_i \ \text{ integer} \\ \pi = \beta & \text{if} \quad \forall i,j,k,l \in [1,4] \, (i \neq j \neq k \neq l) \quad p_i, p_j \ \text{ half-integer}, \quad p_k, p_l \ \text{ integer} \\ \pi = \gamma & \text{if} \quad \forall i \in [1,4] \quad p_i \ \text{ half-integer} \end{cases} . \tag{2.2a}$$

In the case of parity $\boldsymbol{\beta}$, both integer triangles shall be denoted by p, p', both other half-integer by $\overline{p}, \overline{p}'$. The single integer quadrangle is denoted by q, both other half-integer by $\overline{q}, \overline{q}'$.

Here again we leave out our own notations with super triangles $\triangle^S$ for showing compact expressions. Delimiter [ ] around expressions like $q_k - p_i, p_i + \frac{1}{2}, q_k + \frac{1}{2}$ means 'integer part of'. General formula of any $\{6\text{-}j\}^S_\pi$ thus may be written as follows:

$$\begin{Bmatrix} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{Bmatrix}^S_\pi = (-1)^{4\sum\limits_{k=1}^{k=3} J_k j_k} \left[\frac{\prod_{k=1}^{k=3} \prod_{i=1}^{i=4} [q_k - p_i]!}{\prod_{i=1}^{i=4} [p_i + \frac{1}{2}]!} \right]^{\frac{1}{2}} \sum_z \frac{(-1)^z z! \Pi_\pi(z)}{\prod_{i=1}^{i=4} (z - [p_i + \frac{1}{2}])! \prod_{k=1}^{k=3} ([q_k + \frac{1}{2}] - z))!} . \quad (2.2b)$$

Monomials $\Pi_\pi(z)$ are of degree 0 or 1 in z:

$$\Pi_\alpha(z) = 1, \quad (2.2c)$$

$$\Pi_\beta(z) = -z(\overline{q} + \overline{q}' - p - p' + 1) + (\overline{q} + \tfrac{1}{2})(\overline{q}' + \tfrac{1}{2}) - pp' , \quad (2.2d)$$

$$\Pi_\gamma(z) = -z + 2(J_1 j_1 + J_2 j_2 + J_3 j_3) + (J_1 + j_1 + J_2 + j_2 + J_3 + j_3) + \tfrac{1}{2} . \quad (2.2e)$$

As it is obvious for $\pi = \alpha$ or β, all constant terms, factor of z and $\Pi_{\alpha,\beta}(0)$, are integers. For parity γ however, a detailed check is necessary for realizing that $\Pi_\gamma(0)$ is integer also. This is useful for computing $6\text{-}j^S$ numerical values, as we did it with program (`superspins`). `Let us stress that our new expressions (2.1a), (2.2b) in terms of` p `and` q `are far to be an artifact`. Indeed they potentially contain all new properties that will be highlighted in this article.

3 Regge symmetry of a 6-j symbol: a new formulation

It was shown [1] that linear transformations on the spins involved in any classical 6-j symbol lead to additional symmetries, out of the S_4 tetrahedron symmetry.
On the left: the well known expressions, on the right or below: our formulation in terms of q.

$$\begin{Bmatrix} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{Bmatrix} = \begin{Bmatrix} J_1 & \frac{1}{2}(J_2 + J_3 + j_2 - j_3) & \frac{1}{2}(J_2 + J_3 + j_3 - j_2) \\ j_1 & \frac{1}{2}(J_2 + j_3 + j_2 - J_3) & \frac{1}{2}(J_3 + j_3 + j_2 - J_2) \end{Bmatrix} = \begin{Bmatrix} J_1 & \frac{1}{2}q_1 - J_2 & \frac{1}{2}q_1 - J_3 \\ j_1 & \frac{1}{2}q_1 - j_2 & \frac{1}{2}q_1 - j_3 \end{Bmatrix} , \quad (3.1)$$

$$= \begin{Bmatrix} \frac{1}{2}(J_1 + J_3 + j_1 - j_3) & J_2 & \frac{1}{2}(J_1 + J_3 + j_3 - j_1) \\ \frac{1}{2}(j_1 + J_1 + j_3 - J_3) & j_2 & \frac{1}{2}(j_3 + j_1 + J_3 - J_1) \end{Bmatrix} = \begin{Bmatrix} J_2 & \frac{1}{2}q_2 - J_3 & \frac{1}{2}q_2 - J_1 \\ j_2 & \frac{1}{2}q_2 - j_3 & \frac{1}{2}q_2 - j_1 \end{Bmatrix} , \quad (3.2)$$

$$= \begin{Bmatrix} \frac{1}{2}(J_1 + J_2 + j_1 - j_2) & \frac{1}{2}(J_2 + J_1 + j_2 - j_1) & J_3 \\ \frac{1}{2}(j_1 + J_1 + j_2 - J_2) & \frac{1}{2}(j_2 + j_1 + J_2 - J_1) & j_3 \end{Bmatrix} = \begin{Bmatrix} J_3 & \frac{1}{2}q_3 - J_1 & \frac{1}{2}q_3 - J_2 \\ j_3 & \frac{1}{2}q_3 - j_1 & \frac{1}{2}q_3 - j_2 \end{Bmatrix} . \quad (3.3)$$

$$\begin{Bmatrix} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2}(J_2 + j_2 + J_3 - j_3) & \frac{1}{2}(J_1 + j_3 + J_3 - j_1) & \frac{1}{2}(J_1 + j_1 + J_2 - j_2) \\ \frac{1}{2}(J_2 + j_3 + j_2 - J_3) & \frac{1}{2}(J_3 + j_1 + j_3 - J_1) & \frac{1}{2}(J_1 + j_1 + j_2 - J_2) \end{Bmatrix} \quad (3.4)$$

$$= \begin{Bmatrix} \frac{1}{2}q_1 - j_3 & \frac{1}{2}q_2 - j_1 & \frac{1}{2}q_3 - j_2 \\ \frac{1}{2}q_1 - J_3 & \frac{1}{2}q_2 - J_1 & \frac{1}{2}q_3 - J_2 \end{Bmatrix} ,$$

$$= \begin{Bmatrix} \frac{1}{2}(J_2 + j_3 + J_3 - j_2) & \frac{1}{2}(J_1 + j_1 + J_3 - j_3) & \frac{1}{2}(J_1 + j_2 + J_2 - j_1) \\ \frac{1}{2}(J_3 + j_3 + j_2 - J_2) & \frac{1}{2}(J_1 + j_1 + j_3 - J_3) & \frac{1}{2}(J_2 + j_1 + j_2 - J_1) \end{Bmatrix} \quad (3.5)$$

$$= \begin{Bmatrix} \frac{1}{2}q_1 - j_2 & \frac{1}{2}q_2 - j_3 & \frac{1}{2}q_3 - j_1 \\ \frac{1}{2}q_1 - J_2 & \frac{1}{2}q_2 - J_3 & \frac{1}{2}q_3 - J_1 \end{Bmatrix} .$$

By using a cyclic index notation, the transformations can be written under a compact form:

$$\begin{Bmatrix} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{Bmatrix} = \begin{Bmatrix} J_l & \frac{1}{2}q_l - J_m & \frac{1}{2}q_l - J_n \\ j_l & \frac{1}{2}q_l - j_m & \frac{1}{2}q_l - j_n \end{Bmatrix} \quad l \in [1,3] \; l, m, n \text{ cyclic on } [1,2,3]. \quad (3.6)$$

With the same cyclic notations, one may write the remaining Regge transformations as

$$\begin{Bmatrix} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2}q_l - j_m & \frac{1}{2}q_m - j_n & \frac{1}{2}q_n - j_l \\ \frac{1}{2}q_l - J_m & \frac{1}{2}q_m - J_n & \frac{1}{2}q_n - J_l \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2}q_l - j_n & \frac{1}{2}q_m - j_l & \frac{1}{2}q_n - j_m \\ \frac{1}{2}q_l - J_n & \frac{1}{2}q_m - J_l & \frac{1}{2}q_n - J_m \end{Bmatrix} . \quad (3.7)$$

Note some useful relations between triangles and quadrangles:

$$q_l = p_m + p_n - 2j_l\,,\ q_l = p_l + p_4 - 2J_l \ \text{ and } \sum_{k=1}^{k=3} q_k = \sum_{l=1}^{l=4} p_l\,. \tag{3.8}$$

As $2j_l$ is integer any standard 6-j symbol has all triangles p integer, thus any q_l is integer.

Actually alternative formulations of Regge transformations are available. As triangles p and quadrangles q are essential keys of 6-j symbols algebraic formula, we are going to use them for looking exactly how their spins are changed by these transformations. Exactly 12 (non trivial) different algebraic expressions are needed and contained only in the first three equations (3.1-3.3). For each of them, in terms of p or q, one has 3 equivalent formulations. It results in four basic equations written with a cyclic notation on l, m, n cyclic on $[1, 2, 3]$.

$$\tfrac{1}{2}(J_l + J_m + j_l - j_m) = J_l + \tfrac{1}{2}(p_m - p_l) = j_l + \tfrac{1}{2}(p_4 - p_n) = J_m + \tfrac{1}{2}(q_m - q_l), \tag{3.9}$$
$$\tfrac{1}{2}(j_l + j_m + J_l - J_m) = j_l + \tfrac{1}{2}(p_l - p_m) = J_l - \tfrac{1}{2}(p_4 - p_n) = j_m + \tfrac{1}{2}(q_m - q_l), \tag{3.10}$$
$$\tfrac{1}{2}(J_m + J_l + j_m - j_l) = J_m + \tfrac{1}{2}(p_l - p_m) = j_m + \tfrac{1}{2}(p_4 - p_n) = J_l + \tfrac{1}{2}(q_l - q_m), \tag{3.11}$$
$$\tfrac{1}{2}(j_m + j_l + J_m - J_l) = j_m + \tfrac{1}{2}(p_m - p_l) = J_m - \tfrac{1}{2}(p_4 - p_n) = j_l + \tfrac{1}{2}(q_l - q_m). \tag{3.12}$$

3.1 Matrix representation of Regge transformations

Consider a 6-dimensional vector space where any 6-j is a vector $\mathbf{J}$ with 6-components, $J_1, J_2, J_3, j_1, j_2, j_3$, on a canonical basis. Then the five Regge transformations (3.1-3.5) are represented by five 6×6 matrix $\mathbf{R}_\kappa$, $\kappa \in [1, 5]$. according to the following scheme:

$$\mathbf{J}' = \mathbf{R}_\kappa \mathbf{J}. \tag{3.1a}$$

$$\underset{\mathbf{J}}{\begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ j_1 \\ j_2 \\ j_3 \end{bmatrix}},\ \underset{\mathbf{J}'}{\begin{bmatrix} J'_1 \\ J'_2 \\ J'_3 \\ j'_1 \\ j'_2 \\ j'_3 \end{bmatrix}},\ \overset{\mathbf{R}_1}{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}},\ \overset{\mathbf{R}_2}{\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}}, \tag{3.1b}$$

$$\overset{\mathbf{R}_3}{\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}},\ \overset{\mathbf{R}_4}{\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}},\ \overset{\mathbf{R}_5}{\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}}. \tag{3.1c}$$

All these matrices are invertible, diagonalizable and have various properties like

$$(\mathbf{R}_1)^2 = (\mathbf{R}_2)^2 = (\mathbf{R}_3)^2 = I_d, \quad det(\mathbf{R}_1) = det(\mathbf{R}_2) = det(\mathbf{R}_3) = -1. \tag{3.1d}$$

$$\mathbf{R}_1 \times \mathbf{R}_2 = \mathbf{R}_2 \times \mathbf{R}_3 = \mathbf{R}_3 \times \mathbf{R}_1 \quad \text{and} \quad \mathbf{R}_2 \times \mathbf{R}_1 = \mathbf{R}_3 \times \mathbf{R}_2 = \mathbf{R}_1 \times \mathbf{R}_3. \tag{3.1e}$$

$$\mathbf{R}_4 \times \mathbf{R}_5 = \mathbf{R}_5 \times \mathbf{R}_4 = I_d, \quad det(\mathbf{R}_4) = det(\mathbf{R}_5) = 1. \tag{3.1f}$$

$(\,-1,\,1,\,1,\,1,\,1,\,1\,)$ and $\left(\frac{(i\sqrt{3}-1)}{2}, \frac{(i\sqrt{3}-1)}{2}, -\frac{(i\sqrt{3}+1)}{2}, -\frac{(i\sqrt{3}+1)}{2}, 1, 1\right)$ are respectively the eigenvalues of $\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3$ and $\mathbf{R}_4, \mathbf{R}_5$. As said above $\mathbf{R}_4$ or $\mathbf{R}_5$ contains two rows of $\mathbf{R}_1$, two rows of $\mathbf{R}_2$ and two rows of $\mathbf{R}_3$. It can be checked also that all Regge transformations leave invariant the forms $J_1 j_1 + J_2 j_2 + J_3 j_3$ and $J_1 + j_1 + J_2 + j_2 + J_3 + j_3 = \frac{1}{2}\sum_{k=1}^{k=3} q_k\,.$
If symbolically $\|\mathbf{J}\|$ stands for the numerical value of $\mathbf{J} = \left\{ \begin{smallmatrix} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{smallmatrix} \right\}$, one may say that $\|\mathbf{J}\|$ is left invariant by any $\mathbf{R}_\kappa$.

3.2 Features of 6-j symbols generated by Regge transformations

To our knowledge, no real analysis was performed `with a view to algebraic partition`. Analogously with the S_4 tetrahedron permutations, let us introduce some definitions and notations easily understandable. Any 6-j symbol has 24 possible distinct aspects (itself included). This set is denoted by $\mathtt{S}_4$. Each of the five equations (3.1-3.5) is a Regge transformation that we have numbered from 1 to 5 . That can be diagrammed as follows:

$$\{6\text{-}j\} \in \mathtt{S}_4 \xrightarrow{Regge_\kappa} \{6\text{-}j\}^{\mathcal{R}_\kappa} \in \mathtt{S}_4^{\mathcal{R}_\kappa}, \quad \kappa \in [1,5]. \tag{3.2a}$$

$\forall \lambda \neq \kappa$ either sets are the same i.e. $\mathtt{S}_4^{\mathcal{R}_\kappa} \cap \mathtt{S}_4^{\mathcal{R}_\lambda} = \mathtt{S}_4^{\mathcal{R}_\kappa}$, or disjoint i.e. $\mathtt{S}_4^{\mathcal{R}_\kappa} \cap \mathtt{S}_4^{\mathcal{R}_\lambda} = \emptyset$.

A priori, 6 `disjoint` sets should be defined, namely $\mathtt{S}_4(0)$, $\mathtt{S}_4(1)$, $\mathtt{S}_4(2)$, $\mathtt{S}_4(3)$, $\mathtt{S}_4(4)$, $\mathtt{S}_4(5)$.

$$\begin{aligned}
\mathtt{S}_4(0) &= \left\{\{6\text{-}j\} \mid \mathtt{S}_4^{\mathcal{R}_5} = \mathtt{S}_4^{\mathcal{R}_4} = \mathtt{S}_4^{\mathcal{R}_3} = \mathtt{S}_4^{\mathcal{R}_2} = \mathtt{S}_4^{\mathcal{R}_1} = \mathtt{S}_4\right\},\\
\mathtt{S}_4(1) &= \left\{\{6\text{-}j\} \mid \exists \text{ only one } \kappa \mid \mathtt{S}_4^{\mathcal{R}_\kappa} \cap \mathtt{S}_4 = \emptyset\right\},\\
\mathtt{S}_4(2) &= \left\{\{6\text{-}j\} \mid \exists \text{ only two } \kappa, \lambda \mid \mathtt{S}_4^{\mathcal{R}_\kappa} \cap \mathtt{S}_4^{\mathcal{R}_\lambda} \cap \mathtt{S}_4 = \emptyset\right\},\\
\mathtt{S}_4(3) &= \left\{\{6\text{-}j\} \mid \exists \text{ only three } \kappa, \lambda, \mu \mid \mathtt{S}_4^{\mathcal{R}_\kappa} \cap \mathtt{S}_4^{\mathcal{R}_\lambda} \cap \mathtt{S}_4^{\mathcal{R}_\mu} \cap \mathtt{S}_4 = \emptyset\right\},\\
\mathtt{S}_4(4) &= \left\{\{6\text{-}j\} \mid \exists \text{ only four } \kappa, \lambda, \mu, \nu \mid \mathtt{S}_4^{\mathcal{R}_\kappa} \cap \mathtt{S}_4^{\mathcal{R}_\lambda} \cap \mathtt{S}_4^{\mathcal{R}_\mu} \cap \mathtt{S}_4^{\mathcal{R}_\nu} \cap \mathtt{S}_4 = \emptyset\right\},\\
\mathtt{S}_4(5) &= \left\{\{6\text{-}j\} \mid \mathtt{S}_4^{\mathcal{R}_5} \cap \mathtt{S}_4^{\mathcal{R}_4} \cap \mathtt{S}_4^{\mathcal{R}_3} \cap \mathtt{S}_4^{\mathcal{R}_2} \cap \mathtt{S}_4^{\mathcal{R}_1} \cap \mathtt{S}_4 = \emptyset\right\}.
\end{aligned} \tag{3.2b}$$

$\mathcal{R}_{all}$ will denote the set of the five Regge transformations acting on a given $\{6\text{-}j\}$ symbol.

Consider an example related to $\mathtt{S}_4(2)$. $\mathcal{R}_{all}$ applied to $\left\{\begin{smallmatrix} 9 & 8 & 6 \\ 3/2 & 9/2 & 13/2 \end{smallmatrix}\right\}_0$ generates five $\{6\text{-}j\}$:

$$\left\{\begin{smallmatrix} 9 & 9/2 & 13/2 \\ 3/2 & 8 & 6 \end{smallmatrix}\right\}^{\mathcal{R}_1}, \left\{\begin{smallmatrix} 8 & 11/2 & 5/2 \\ 9/2 & 5 & 10 \end{smallmatrix}\right\}^{\mathcal{R}_2}, \left\{\begin{smallmatrix} 6 & 5/2 & 7/2 \\ 13/2 & 10 & 7 \end{smallmatrix}\right\}^{\mathcal{R}_3}, \left\{\begin{smallmatrix} 6 & 10 & 7 \\ 13/2 & 5/2 & 7/2 \end{smallmatrix}\right\}^{\mathcal{R}_4}, \left\{\begin{smallmatrix} 8 & 5 & 10 \\ 9/2 & 11/2 & 5/2 \end{smallmatrix}\right\}^{\mathcal{R}_5}. \tag{3.2c}$$

As $\exists\, s \in S_4 \,|\, \left\{\begin{smallmatrix} 9 & 9/2 & 13/2 \\ 3/2 & 8 & 6 \end{smallmatrix}\right\}^{\mathcal{R}_1} \xrightarrow{s} \left\{\begin{smallmatrix} 9 & 8 & 6 \\ 3/2 & 9/2 & 13/2 \end{smallmatrix}\right\}_0$, we remove $\cancel{\left\{\begin{smallmatrix} 9 & 9/2 & 13/2 \\ 3/2 & 8 & 6 \end{smallmatrix}\right\}^{\mathcal{R}_1}}$ from the list (3.2c), the same holds for $\cancel{\left\{\begin{smallmatrix} 8 & 5 & 10 \\ 9/2 & 11/2 & 5/2 \end{smallmatrix}\right\}^{\mathcal{R}_5}}$ with $\left\{\begin{smallmatrix} 8 & 11/2 & 5/2 \\ 9/2 & 5 & 10 \end{smallmatrix}\right\}^{\mathcal{R}_2}$, and $\cancel{\left\{\begin{smallmatrix} 6 & 10 & 7 \\ 13/2 & 5/2 & 7/2 \end{smallmatrix}\right\}^{\mathcal{R}_4}}$ with $\left\{\begin{smallmatrix} 6 & 5/2 & 7/2 \\ 13/2 & 10 & 7 \end{smallmatrix}\right\}^{\mathcal{R}_3}$. Thus, after last convenient[1] S_4 rearrangements, it remains only three representative $\{6\text{-}j\}$ of the set $\mathtt{S}_4(2)$ in our example, numbered from 0 to 2:

$$\left\{\begin{smallmatrix} 9 & 8 & 6 \\ 3/2 & 9/2 & 13/2 \end{smallmatrix}\right\}_0 \;\&\; \left\{\begin{smallmatrix} 10 & 8 & 5 \\ 5/2 & 9/2 & 11/2 \end{smallmatrix}\right\}_1^{(\mathcal{R}_2)} \;\&\; \left\{\begin{smallmatrix} 10 & 7 & 6 \\ 5/2 & 7/2 & 13/2 \end{smallmatrix}\right\}_2^{(\mathcal{R}_3)} = -\tfrac{1}{2}\sqrt{\tfrac{23}{5\times7\times13\times17}}\,. \tag{3.2d}$$

The operation realized for reducing the list (3.2c) acts as a filter that we denote by (S_4 filter).

`Definition` of $\mathcal{R}^*_{egge}$:

$$\mathcal{R}^*_{egge} = (S_4\,\text{filter}) \circ \mathcal{R}_{all}. \tag{3.2e}$$

More generally, $\forall\, \{6\text{-}j\}_0 \in \mathtt{S}_4(2)$, $\mathcal{R}^*_{egge}$ generates two 'different' $\{6\text{-}j\}$ symbols, namely $\{6\text{-}j\}_1$ and $\{6\text{-}j\}_2$. Numerical values are equal, but their triangles are not the same. One has:

$$\begin{aligned}
\{6\text{-}j\}_0 &\xrightarrow{\mathcal{R}^*_{egge}} \{6\text{-}j\}_1 \;\&\; \{6\text{-}j\}_2,\\
\{6\text{-}j\}_1 &\xrightarrow{\mathcal{R}^*_{egge}} \{6\text{-}j\}_0 \;\&\; \{6\text{-}j\}_2,\\
\{6\text{-}j\}_2 &\xrightarrow{\mathcal{R}^*_{egge}} \{6\text{-}j\}_0 \;\&\; \{6\text{-}j\}_1.
\end{aligned} \tag{3.2f}$$

[1] means here with spins arranged according to the entries listed in Table like that of Rotenberg and al. [6].

The order in right equation members above is not relevant, one can take $1, 2$ or $2, 1$ and so on. Clearly the set $\mathtt{S}_4(2)$ is `stable` under $\mathcal{R}^*_{egge}$. The number of its elements is $\mathrm{Card}(\mathtt{S}_4(2))$ i.e. $\#(\mathtt{S}_4(2)) = 3 \times 24 = 72$. Now the main task is to discover what are the parameters which define the presumed partitions $\mathtt{S}_4(0) \cdots \mathtt{S}_4(5)$.

`After solving the partitions, the results are summarized below:`

Here we implicitly assume that $l \in [1, 3]$ and l, m, n cyclic on $[1, 2, 3]$, even if there is another index like l'. A notation like (all $p \neq$) means that for the full set of p_1, p_2, p_3, p_4 all p are different between them.

$$\mathtt{S}_4(0) = \big\{\{6\text{-}j\}\big\} \mid \begin{array}{c} (\text{all } q \text{ equal }) \\ \mathtt{or} \\ (q_3 \neq q_1)\ \mathtt{and}\ ((\text{all } p \text{ equal})\ \mathtt{or}\ (\text{all } p_l \text{ equal} \neq p_4)) \\ \mathtt{or} \\ (q_3 \neq q_1)\ \mathtt{and}\ (p_1 = p_2 = p_4 \neq p_3) \end{array}. \tag{3.2g}$$

$$\mathtt{S}_4(1) = \big\{\{6\text{-}j\}\big\} \mid \begin{array}{c} (q_l = q_m \neq q_n)\ \mathtt{and}\ (\text{all } p_{l'} \neq p_4)\ \mathtt{and}\ (p_{l'} = p_{m'} \neq p_{n'}) \\ \mathtt{or} \\ (q_l = q_m \neq q_n)\ \mathtt{and}\ (\text{all } \ p_{l'} \neq)\ \mathtt{and}\ ((p_1 = p_4)\ \mathtt{or}\ (p_2 = p_4)) \\ \mathtt{or} \\ (q_l = q_m \neq q_n)\ \mathtt{and}\ (p_1 \neq p_2)\ \mathtt{and}\ (p_2 = p_3)\ \mathtt{and}\ (p_1 = p_4) \\ \mathtt{or} \\ (\ q_2 = q_3 \neq q_1)\ \mathtt{and}\ (p_1 \neq p_2)\ \mathtt{and}\ (p_1 = p_3)\ \mathtt{and}\ (p_2 = p_4) \\ \mathtt{or} \\ (\ q_1 = q_2 \neq q_3)\ \mathtt{and}\ (p_1 = p_2 \neq p_3)\ \mathtt{and}\ (p_3 = p_4) \end{array}. \tag{3.2h}$$

$$\mathtt{S}_4(2) = \big\{\{6\text{-}j\}\big\} \mid \begin{array}{c} (q_l = q_m)\ \mathtt{and}\ (\text{all } p \neq) \\ \mathtt{or} \\ (\text{all } q \neq)\ \mathtt{and}\ (p_l = p_m)\ \mathtt{and}\ (\text{all } p_l \neq p_4) \\ \mathtt{or} \\ (\text{all } q \neq)\ \mathtt{and}\ (p_l = p_m)\ \mathtt{and}\ (p_n = p_4) \\ \mathtt{or} \\ (\text{all } q \neq)\ \mathtt{and}\ (\text{all } p_l \neq)\ \mathtt{and}\ ((p_1 = p_4)\ \mathtt{or}\ (p_2 = p_4)) \end{array}. \tag{3.2i}$$

$$\mathtt{S}_4(3) = \emptyset\,. \tag{3.2j}$$

$$\mathtt{S}_4(4) = \emptyset\,. \tag{3.2k}$$

$$\mathtt{S}_4(5) = \big\{\{6\text{-}j\}\big\} \mid (\text{all } q \neq)\ \mathtt{and}\ (\text{all } p \neq). \tag{3.2l}$$

$\mathtt{S}_4(0)$ is `stable` under $\mathcal{R}^*_{egge}$. No new triangle is created, consequence of (3.9-3.12). Actually each of $\mathtt{S}_4(1), \mathtt{S}_4(2), \mathtt{S}_4(5)$ are `stable` also. Our notion of 'stability' for these sets has been well defined. It is known also that [example of the matrices $\mathbf{R}_\kappa$] any eigenspace is stable under $\mathbf{R}_\kappa$. Is there a correlation between these eigenspaces and sets $\mathtt{S}_4(\,)$? It would be worth to deepen in this topic.

Moreover why our initial partitions imply that $\mathtt{S}_4(3)$ and $\mathtt{S}_4(4)$ are empty sets arises from our natural definitions (3.2b) and the fact that additional constraint of inequalities to (3.2i) would lead to $\mathtt{S}_4(3) \rightarrow \mathtt{S}_4(4) \Rightarrow \mathtt{S}_4(3) \equiv \mathtt{S}_4(4) \rightarrow \mathtt{S}_4(5) \Rightarrow \mathtt{S}_4(3) \equiv \mathtt{S}_4(4) \equiv \mathtt{S}_4(5)$. Then $\mathtt{S}_4(3), \mathtt{S}_4(4)$ cannot exist. That can be understood only after solving the algebraic partitions.

Initial identification of algebraic features of $\mathtt{S}_4(0)$ was easily found by hand calculation, however, for the remaining sets, it becomes less obvious. That is why it was decided to write a program (`symmetryregge`) called for help, comparing a lot of $\mathtt{S}_4$ sets. The process is not fully automatic and requires careful analysis. Originally scheduled for 6-j^S super-symbols, it turns out that it obviously applies to standard 6-j symbols, where no distinction has to be made between parities α, β, γ. In this case we only have to handle α.
Our logical partitions of the disjoint subsets themselves, i.e. by using `'and'` and `'or'`, are quite right, but maybe not necessarily the best, because a general symmetry does not appear since sometimes q_1, q_3, p_4 are singularized. It should be possible to find other rearrangements.

Now all features of standard 6-j symbols can be symbolically summarized by a sequence, where over each subset is indicated its cardinal #:

$$\{6\text{-}j\} \xrightarrow{\mathcal{R}^*_{egge}} \overset{\#=24}{\mathtt{S}_4(0)} \oplus \overset{\#=48}{\mathtt{S}_4(1)} \oplus \overset{\#=72}{\mathtt{S}_4(2)} \oplus \overset{\#=144}{\mathtt{S}_4(5)}\ . \tag{3.2m}$$

Thus adding Regge symmetries to standard 6-j symbols changes their symmetry groups according to the subset to which they belong. The table below shows the related isomorphisms.

$\{6\text{-}j\}$ + Regge symmetry (3.2n)

$\mathtt{S}_4(0)$	$\mathtt{S}_4(1)$	$\mathtt{S}_4(2)$	$\mathtt{S}_4(5)$
S_4	$S_4 \times S_2$	$S_4 \times (S_3/S_2)$	$S_4 \times S_3$

4 Regge symmetry of 6-j^S symbols

In 1992 was published a preliminary study dealing with this subject [7] and 'appended' inside the paper which followed [4]. The authors then were able to announce unexpected results namely: a similar Regge symmetry exists also for 6-j^S symbols [4], but sometimes smaller than in the case of $so(3)$ [7]. Their proof, self-qualified of *"laborious"* was omitted. Results were presented by using phase factors, with inference to Bargmann's representation [8].

As since 1993, nothing was known about partition properties, it seemed interesting to further develop this topic also for 6-j^S symbols, as a complement to our previous work [5].
In the present case, it will be seen that use of parities $\pi = \alpha, \beta, \gamma$ and q formulation provides clear and immediate solutions, `free of explicit phase factors and without need of another representation.` General results cited just above [7, 4] are significantly better understood by means of our approach with partitions. Moreover it brings unknown information related to p and q.

First of all, in virtue of our analytic formulas for 6-j^S symbols, (2.2b) compared to (2.1a), clearly one can assert that a 6-j^S, as well as a standard 6-j, possesses the same symmetry S_4. This property can thus be used as desired.
In this section, the only difference lies in the fact that triangles p may be integer or half-integer as well as quadrangles q. Depending on parities $\pi = \alpha, \beta, \gamma$, one has the following properties:
$\pi = \alpha$ or $\gamma \Rightarrow$ all q are integers.
$\pi = \beta \Rightarrow \exists!\, q_{l^*}$ integer (the other two are half-integer). l^* is a distinguished index.
As any q half-integer is excluded [2] from the Regge transformations the conclusion is clear:
Case $\boldsymbol{\alpha}$ or $\boldsymbol{\gamma}$: all Regge symmetries are valid.
Case $\boldsymbol{\beta}$: a single transformation is valid.

[2]Spins like $(2n+1)/4$ forbidden.

After noting that parity π remains invariant under any Regge transformation, one can write:

$$\begin{Bmatrix} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{Bmatrix}^S_{\boldsymbol{\alpha},\boldsymbol{\gamma}} = \begin{Bmatrix} J_l & \frac{1}{2}q_l - J_m & \frac{1}{2}q_l - J_n \\ j_l & \frac{1}{2}q_l - j_m & \frac{1}{2}q_l - j_n \end{Bmatrix}^S_{\boldsymbol{\alpha},\boldsymbol{\gamma}} \qquad l \in [1,3]\ l,m,n \text{ cyclic on } [1,2,3], \tag{4.1}$$

$$\begin{Bmatrix} J_1 & J_2 & J_3 \\ j_1 & j_2 & j_3 \end{Bmatrix}^S_{\boldsymbol{\alpha},\boldsymbol{\gamma}} = \begin{Bmatrix} \frac{1}{2}q_l - j_m & \frac{1}{2}q_m - j_n & \frac{1}{2}q_n - j_l \\ \frac{1}{2}q_l - J_m & \frac{1}{2}q_m - J_n & \frac{1}{2}q_n - J_l \end{Bmatrix}^S_{\boldsymbol{\alpha},\boldsymbol{\gamma}} = \begin{Bmatrix} \frac{1}{2}q_l - j_n & \frac{1}{2}q_m - j_l & \frac{1}{2}q_n - j_m \\ \frac{1}{2}q_l - J_n & \frac{1}{2}q_m - J_l & \frac{1}{2}q_n - J_m \end{Bmatrix}^S_{\boldsymbol{\alpha},\boldsymbol{\gamma}}. \tag{4.2}$$

$$\begin{Bmatrix} J_{l^*} & J_m & J_n \\ j_{l^*} & j_m & j_n \end{Bmatrix}^S_{\boldsymbol{\beta}} = \begin{Bmatrix} J_{l^*} & \frac{1}{2}q_{l^*} - J_m & \frac{1}{2}q_{l^*} - J_n \\ j_{l^*} & \frac{1}{2}q_{l^*} - j_m & \frac{1}{2}q_{l^*} - j_n \end{Bmatrix}^S_{\boldsymbol{\beta}} \qquad \text{(only one relation for } \boldsymbol{\beta}). \tag{4.3}$$

4.1 Features of 6-j^S symbols generated by Regge transformations

It is obvious that, for parity $\boldsymbol{\alpha}$ or $\boldsymbol{\gamma}$, properties like (3.2g-3.2l) are still valid. Thus

$$\{6\text{-}j\}^S_{\alpha,\gamma} \xrightarrow{\mathcal{R}^*_{egge}} \overset{\#=24}{\mathtt{S}^S_4(0)} \oplus \overset{\#=48}{\mathtt{S}^S_4(1)} \oplus \overset{\#=72}{\mathtt{S}^S_4(2)} \oplus \overset{\#=144}{\mathtt{S}^S_4(5)}\ . \tag{4.1a}$$

However for parity $\pi = \boldsymbol{\beta}$, only sets like $\mathtt{S}^S_4(0)$ or $\mathtt{S}^S_4(1)$ can exist. They are defined by

$$\mathtt{S}^S_4(0) = \left\{\{6\text{-}j\}^S\right\} \mid (\overline{q} = \overline{q}')\ \mathtt{or}\ ((\overline{q} \neq \overline{q}')\ \mathtt{and}\ ((p = p')\ \mathtt{or}\ (\overline{p} = \overline{p}'))). \tag{4.1b}$$

$$\mathtt{S}^S_4(1) = \left\{\{6\text{-}j\}^S\right\} \mid (\overline{q} \neq \overline{q}')\ \mathtt{and}\ (p \neq p')\ \mathtt{and}\ (\overline{p} \neq \overline{p}'). \tag{4.1c}$$

One can then write

$$\{6\text{-}j\}^S_{\beta} \xrightarrow{\mathcal{R}^*_{egge}} \overset{\#=24}{\mathtt{S}^S_4(0)} \oplus \overset{\#=48}{\mathtt{S}^S_4(1)}\ . \tag{4.1d}$$

Analogous isomorphism tables to (3.2n) are as follows:

$\{6\text{-}j\}^S_{\alpha,\gamma}$ + Regge symmetry (4.1e)

$\mathtt{S}^S_4(0)$	$\mathtt{S}^S_4(1)$	$\mathtt{S}^S_4(2)$	$\mathtt{S}^S_4(5)$
S_4	$S_4 \times S_2$	$S_4 \times (S_3/S_2)$	$S_4 \times S_3$

and

$\{6\text{-}j\}^S_{\beta}$ + Regge symmetry (4.1f)

$\mathtt{S}^S_4(0)$	$\mathtt{S}^S_4(1)$
S_4	$S_4 \times S_2$

Our conclusion does not differ appreciably from that of Daumens *et al.* [7, 4], but is much more accurate with unknown information that we have highlighted.

5 Program notes

Mathematical properties detailed in **2.2** has made easier the coding of a computing program (`superspins`), written in fortran 95. As a result, editing the table of 6-j^S symbols, from $\begin{Bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix}^S$ to $\begin{Bmatrix} 10 & 10 & 10 \\ 10 & 10 & 10 \end{Bmatrix}^S$ takes only 6 seconds with a small home computer. Of course this was processed in the same spirit than that used for producing the famous Rotenberg's tables [6]. All is computed with integers, factorial and prime numbers. Redundancies were avoided as far as possible. As expected, the fit with the reduced data published in Ref. [4] is perfect.

We have enlarged the basis of prime numbers: 2 to 31 becomes 2 to 53, namely $\mathtt{p_1}$ to $\mathtt{p_{16}}$. Program (`superspins`) computes their exponents $\mathtt{e_i}, \mathtt{i} \in [1, 16]$. You have to evaluate the expression- generally fractional- $\prod_{i=1}^{16} (\mathtt{p_i})^{\mathtt{e_i}}$, take its square root and multiply by the integer which follows the ampersand &. If beyond 53, this integer is showed under the form of a prime numbers product. Then you obtain the exact numerical value of a 6-j^S symbol.
In addition, the parity is listed as <a>, <b>, <g> respectively for α, β, γ . Excerpts of the table are given below.

```
--> super 6JS symbols         |       exponents of the 16 first prime        | | signed integer |
2    2    3/2  | 2    1    3/2  | -2  0 -2  2  0  0  0  0  0  0  0  0  0  0  0  0  &                 -1 <g>
2    2    2    | 2    2    3/2  | -2  2 -2  0  0  0  0  0  0  0  0  0  0  0  0  0  &                 -1 <b>
2    2    2    | 2    2    2    | -2  2 -2  0  0  0  0  0  0  0  0  0  0  0  0  0  &                 -1 <a>
4    7/2  7/2  | 7/2  3    3    |  2 -1 -4 -2  1  0  0  0  0  0  0  0  0  0  0  0  &                  0 <a>
11/2 5    9/2  | 4    3    7/2  |  0 -2  3 -1 -2  0  0  0  0  0  0  0  0  0  0  0  &                  1 <b>
13/2 13/2 4    | 9/2  9/2  6    | -3 -2 -1 -2 -2 -2 -1  0  0  0  0  0  0  0  0  0  &              13537 <a>
7    11/2 4    | 6    11/2 5    |  1  0 -1 -1 -2 -2 -1  0  0  0  0  0  0  0  0  0  &              -1493 <g>
7    11/2 9/2  | 5/2  7/2  5    | -3 -2 -2  1 -2 -1  1  0  0  0  0  0  0  0  0  2  &                  1 <b>
19/2 19/2 17/2 | 19/2 9    17/2 | -2 -6 -4  0 -4 -4 -2 -2 -2  2  0  0  0  0  0  0  & -(1129x35248061) <b>
10   17/2 11/2 | 3/2  5    8    | -3 -2 -1  0 -1  0 -1  0  0  0  0  0  0  0  0  0  &                107 <a>
10   17/2 8    | 4    13/2 6    | -7 -1  3 -2  0 -1 -2 -2  0  0  0  0  0  0  0  0  &               8081 <g>
10   10   9    | 10   10   9    | -6  0 -4 -2 -2  0 -2 -2 -2 -2  0  0  0  0  0  0  &  -(61x241x53117) <a>
10   10   17/2 | 9    17/2 10   | -5 -1 -1 -1 -1  0 -2 -2 -2 -1  0  0  0  0  0  0  &       (571x1951) <b>
10   10   17/2 | 10   10   17/2 | -6 -2 -4 -2 -2  0 -2 -2 -2 -2  0  0  0  0  0  0  & (157x1583x13043) <g>
```

Output files (format .txt):
`superspinstest,superspzeroa,superspzerob,superspzerog,`
`symmreggetesta0,symmreggetesta1,symmreggetesta2,symmreggetesta5,`
`symmreggetestb0,symmreggetestb1,`
`symmreggetestg0,symmreggetestg1,symmreggetestg2,symmreggetestg5.`
The first contains the full table, the other file names suggest their contents.
Of course a slight reduction and change of (`superspins`)$\rightarrow$ (`standardsymbols`) allows us to recover instantly the part of Rotenberg's table [6] which lists the standard 6-j symbols.

6 Conclusion

Contrary to received ideas, first of all it seems improper to assert that adding Regge symmetry to any 6-j symbols automatically gives them a symmetry group isomorphic to $S_4 \times S_3$, because subsets like $\mathtt{S_4(0)}, \mathtt{S_4(1)}, \mathtt{S_4(2)}$ are quite well defined, according to Table (3.2n). The same holds for any super 6-j^S symbols with the subsets $\mathtt{S}_4^S(0), \mathtt{S}_4^S(1), \mathtt{S}_4^S(2)$, according to Tables (4.1e),(4.1f). In the same way the assertion that *"the symmetry group of an arbitrary* 6-j^S *symbol contains at least* 48 *elements"* [4] should be re-formulated.
In summary the possible isomorphic symmetry groups are the following:

$$
\begin{array}{c}
\{6\text{-}j\} \text{ and } \{6\text{-}j\}^S_{\alpha,\gamma} \\
\begin{array}{|c|c|c|c|} \hline S_4 & S_4 \times S_2 & S_4 \times (S_3/S_2) & S_4 \times S_3 \\ \hline \end{array}
\end{array}
\qquad
\begin{array}{c}
\{6\text{-}j\}^S_{\beta} \\
\begin{array}{|c|c|} \hline S_4 & S_4 \times S_2 \\ \hline \end{array}
\end{array}
\tag{4.1g}
$$

This study made us discover a hidden classification of each 6-j symbol, according to their belonging to four disjoint sets. A new notation could be ${}_c\{6\text{-}j\}$ where $c = 0, 1, 2, 5$. These symbols are directly evidenced in spectroscopic experiments `in nuclear, atomic or molecular physics`, generally for evaluating transition intensities or interaction energy. As a first step, a review of the type of implied symbols could reveal whether parameter c is significant or not.

References

[1] T. Regge, Symmetry properties of Racah's coefficients, Lettere al Nuovo Cimento, **11** (1959), 116-117.

[2] L.C. Biedenharn and J.D. Louck, Angular Momentum in Quantum Physics, Theory and Application, Encyclopedia of Mathematics and its Applications, ed., Addison-Wesley, London, 1981, Vol. 8, pp. 124-125.

[3] A.R. Edmonds, Angular Momentum in Quantum Mechanics, ed., Princeton U.P., Princeton N.J., 1974.

[4] M. Daumens, P. Minnaert, M. Mozrzymas and S. Toshev, The super-rotation Racah-Wigner calculus revisited, J. Math. Phys., **34** (1993), 2475-2507.

[5] L. Bréhamet, Analytical complements to the parity-independent Racah-Wigner calculus for the superalgebra osp(1|2). Part I, Nuovo Cimento, **B121** (2006), 241-274.

[6] M. Rotenberg, R. Bivins, N. Metropolis and J.K. Jr. Wooten, The 3-j and 6-j symbols, ed., Cambridge MIT Press, 1959.

[7] M. Daumens, P. Minnaert, M. Mozrzymas and S. Toshev, Regge Symmetries of Super 6-j Symbols for the Superalgebra osp(1|2), Europhys. Lett., **20** (1992), 671-676.

[8] V. Bargmann, On the Representations of the Rotation Group, Rev. Mod. Phys., **34** (1962), 829-845.